Vibration of Cylindrical Gridwork Shells

THEIN WAH*
Southwest Research Institute, San Antonio, Texas

The natural frequencies of a circular cylindrical gridwork shell, consisting of a system of equally spaced rings and longitudinal stringers, are investigated by use of finite-difference calculus. A solution is derived for the case of simply supported boundaries.

Nomenclature

a = radius of ring
b = spacing of rings
ϕ = circumferential coordinate
u, v, w = axial, circumferential, and radial displacement, respectively
u_{rs} = $U_{rs} \exp i\omega t$ }
v_{rs} = $V_{rs} \exp i\omega t$ } axial, circumferential, and radial displacements at the node (r, s)
w_{rs} = $W_{rs} \exp i\omega t$ }
Q_{rs}, Q_{rs}' = shearing force in the plane and at right angles to the plane of the ring
$\bar{Q}_{rs}, \bar{Q}_{rs}'$ = shearing forces on a stringer in two perpendicular directions
M_{rs}, M_{rs}' = bending moments on a ring section at (r, s)
$\bar{M}_{rs}, \bar{M}_{rs}'$ = bending moments on a stringer at (r, s)
N_{rs} = axial tension on a ring at (r, s)
$\bar{N}_{rs}$ = axial tension on a stringer at (r, s)
E = Young's modulus of elasticity
$\mathbf{E}$ = shifting operator
G = shear modulus
$T_{rs}, \bar{T}_{rs}$ = twisting moment in stringer and ring, respectively
ρ = volume density
ρ^* = concentrated mass at each node
I, I_ϕ = moments of inertia of ring
$\bar{I}, \bar{I}_\phi$ = moments of inertia of stringer
I_p = polar moment of inertia of ring
$\bar{I}_p$ = polar moment of inertia of stringer
J = torsion constant for ring
Θ = rotation of ring section
$\bar{\Theta}, \Theta'$ = rotation of stringer section in two perpendicular directions
$\bar{J}$ = torsion constant for stringer
A = area of cross section of ring
$\bar{A}$ = area of cross section of stringer
n = number of circumferential lobes
m = number of half-waves in the axial direction
R = number of stringers
S = number of spacings between rings
(r, s) = generic node
t = time
p = $(\rho\omega^2\alpha^4a^4A/EI_\phi)^{1/4}$
$\bar{p}$ = $(\rho\omega^2b^4\bar{A}/E\bar{I})^{1/4}$
p' = $(\rho\omega^2b^4\bar{A}/E\bar{I}_\phi)^{1/4}$
q = $\alpha[(\rho\omega^2a^2\alpha^2I_p/GJ) - (EI_\phi/GJ)]^{1/2}$
l = $(\rho\omega^2b^2\bar{I}_p/G\bar{J})^{1/2}$
j = $(\rho\omega^2b^2/E)^{1/2}$
K = $(\rho\omega^2a^4A/EI)^{1/2}$
$c_1, c_2, \ldots, c_{12}$ = defined by Eqs. (1) and (A17) }
c_{13}, c_{14} = defined by Eq. (4) }
c_{15}–c_{20}, $\bar{c}_{15}$–$\bar{c}_{20}$, c_{15}'–c_{20}' = defined by Eqs. (6, 8, and 9) } For distributed mass solution
c_{21}, c_{22} = defined by Eq. (11) }
c_{23}, c_{24} = defined by Eq. (13) }
c_1–c_{24} = defined by Eqs. (26) and (27) for lumped mass solution
$\lambda, \lambda_1, \lambda_2$, ν, ν_1, μ, μ_1, ϵ = defined by Eq. (18)
γ_1–γ_5 = defined by Eq. (24)
$\bar{\gamma}_1$–$\bar{\gamma}_5$ = defined by Eq. (28)
α = angular spacing of stringers

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* Staff Scientist.

Introduction

In most practical applications, thin cylindrical shells need to be stiffened in some manner. In evaluating the dynamic characteristics of a shell, therefore, one has usually to take into account the effect of the stiffening members. Generally, cylindrical shells are stiffened in either of two ways: by means of equally spaced ring frames, or by both ring frames and longitudinal stringers running parallel to the shell axis. It is the latter type of stiffening that is our concern in this paper.

The common and generally accepted approach for a stiffened cylindrical shell is to treat it as an orthotropic shell. This, in effect, involves the approximating assumption that the stiffening system has a continuous distribution.

Another and different viewpoint is certainly possible. This is to fix attention on the stiffening system consisting of rings and stringers and to treat this as the basic structure. The effect of the skin may be approximated in some fashion, e.g., by including a portion of it on either side of a ring or stringer in calculating the geometrical property of those elements. In those not infrequently occurring cases where the stiffening system is widely spaced and is considerably heavier than the shell plating, this indeed would be the logical approximation. The present paper is an attempt to obtain a theoretical solution for such a gridwork shell.

In the analysis, the skin is assumed to be absent. The gridwork shell is idealized as shown in Fig. 1, the rings and stringers being supposed rigidly connected at the intersections. This idealization is likely to be a good one for those modes involving primarily w displacements. When the

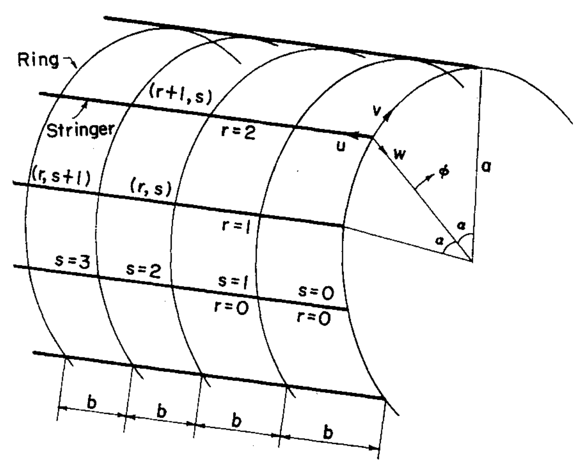

Fig. 1 Definition sketch for gridwork shells.

modes are dominantly u and v displacements, the shear stiffness of the shell skin could make the idealization a much less accurate representation of the stiffened shell.

Finite-difference calculus is used to investigate the vibrations of this cylindrical gridwork.† An approximate formulation, also using finite-difference calculus, is developed later, on the assumption that the mass of the rings and stringers may be assumed concentrated equally at the intersections. This formulation has the advantage that the natural frequencies of a simply supported shell may be obtained explicitly from a cubic equation, whereas the more exact approach requires the solution of a transcendental equation of some complexity.

In order to render the mathematical solution reasonably simple, certain assumptions have been made. It is supposed that the cross sections of the equally spaced and identical rings and stringers are such that the flexural and torsional vibrations are uncoupled. Further, the purely tangential vibrations of the rings are neglected.

Equations of Motion

The coordinate system used is shown in Fig. 1. There are assumed to be R stringers equally spaced at $r = 0, 1, 2, \ldots, R - 1$ along the circumference of a ring. The mean radius of a ring is denoted by a, and the rings are placed at $s = 0, 1, 2, \ldots, S$.

Consider first the flexural vibrations in the plane of the ring. The forces acting at the extremities of the ring segment extending from (r, s) to $(r + 1, s)$ are shown in Fig. 2a and the corresponding displacements in Fig. 2b. The superscripts R and L denote right and left of the node under consideration.

It is shown in the Appendix that the relations between the forces and displacements may be written as follows:

$$\left.\begin{aligned}
-N_{rs}{}^R &= (EI/\alpha^3a^3)(c_1V_{rs} + c_2W_{rs} + c_3\alpha a\Theta_{rs} + \\
&\qquad c_4V_{r+1,\,s} + c_2W_{r+1,\,s} + c_6\alpha a\Theta_{r+1,\,s}) \\
-Q_{rs}{}^R &= (EI/\alpha^3a^3)(c_2V_{rs} + c_7W_{rs} + c_8\alpha a\Theta_{rs} - \\
&\qquad c_5V_{r+1,\,s} + c_9W_{r+1,\,s} + c_{10}\alpha a\Theta_{r+1,\,s}) \\
M_{rs}{}^R &= (EI/\alpha^2a^2)(c_3V_{rs} + c_8W_{rs} + c_{11}\alpha a\Theta_{rs} + \\
&\qquad c_6V_{r+1,\,s} - c_{10}W_{r+1,\,s} + c_{12}\alpha a\Theta_{r+1,\,s}) \\
N_{r+1,\,s}{}^L &= (EI/\alpha^3a^3)(c_4V_{rs} - c_5W_{rs} + \\
&\qquad c_6\alpha a\Theta_{rs} + c_1V_{r+1,\,s} - c_2W_{r+1,\,s} + c_3\alpha a\Theta_{r+1,\,s}) \\
Q_{r+1,\,s}{}^L &= (EI/\alpha^3a^3)(c_5V_{rs} + c_9W_{rs} - c_{10}\alpha a\Theta_{rs} - \\
&\qquad c_2V_{r+1,\,s} + c_7W_{r+1,\,s} - c_8\alpha a\Theta_{r+1,\,s}) \\
-M_{r+1,\,s}{}^L &= (EI/\alpha^2a^2)(c_6V_{rs} + c_{10}W_{rs} + c_{12}\alpha a\Theta_{rs} + \\
&\qquad c_3V_{r+1,\,s} - c_8W_{r+1,\,s} + c_{11}\alpha a\Theta_{r+1,\,s})
\end{aligned}\right\} \quad (1)$$

In Eqs. (1) and the equations that follow, the time function $\exp(i\omega t)$, where ω is the circular frequency, is omitted from both sides. The coefficients c_i are functions of the frequency ω, the mass density ρ, and the geometrical properties of the ring. They are defined in the Appendix. N, Q, and M stand for the axial tension, shearing force, and bending moment, respectively. V, W, and Θ are the amplitude of the tangential displacement, inward radial displacement, and rotation, respectively.

Considering next the torsional vibrations of the ring, the torque $\bar{T}_{rs}$ and the transverse shears Q_{rs}' at right angles to the plane of the ring are given as follows (see Appendix and Fig. 3):

$$\bar{T}_{rs} = -(GJ/\alpha a)[c_{14}(\bar{\Theta}_{r+1,\,s} + \bar{\Theta}_{r-1,\,s}) - 2c_{13}\bar{\Theta}_{rs}] \quad (2)$$

$$Q_{rs}' = -(\epsilon GJ/a^2\alpha)[c_{14}(\bar{\Theta}_{r+1,\,s} + \bar{\Theta}_{r-1,\,s}) - 2c_{13}\bar{\Theta}_{rs}] \quad (3)$$

† This method, despite formal resemblances, has no necessary elation to matrix methods of analysis.

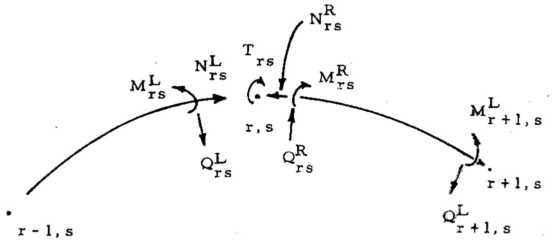

Fig. 2a Forces on ring segment (flexure).

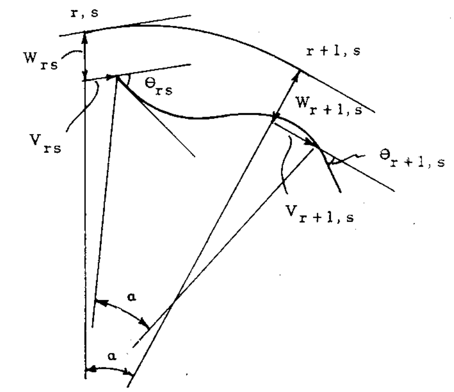

Fig. 2b Displacements of ring segment (flexure).

with

$$c_{13} = q\cot q \qquad c_{14} = q\,\mathrm{cosec}\,q \qquad \epsilon = (1 + EI_\phi/GJ)$$

$$q = \alpha[(a^2I_p\rho\omega^2/GJ) - (EI_\phi/GJ)]^{1/2} \quad (4)$$

G is the shear modulus, J the torsion constant for the ring section, ρ the mass density, I_p the polar moment of inertia, and I_ϕ the moment of inertia about an axis in the plane of the ring. $\bar{\Theta}_{rs}$ is the angle of twist at (r, s), considered positive in the positive direction of the s coordinate (Fig. 1).

The flexural vibrations of the ring at right angles to its plane give the relations (see Appendix)

$$\left.\begin{aligned}
-Q_{rs}{}'^R &= (EI_\phi/\alpha^3a^3)(c_{15}U_{rs} + c_{16}a\alpha\Theta_{rs}' - \\
&\qquad c_{17}U_{r+1,\,s} + c_{18}a\alpha\Theta_{r+1,\,s}') \\
M_{rs}{}'^R &= (EI_\phi/\alpha^2a^2)(c_{16}U_{rs} + c_{19}a\alpha\Theta_{rs}' - \\
&\qquad c_{18}U_{r+1,\,s} + c_{20}a\alpha\Theta_{r+1,\,s}') \\
Q_{r+1,\,s}{}'^L &= (EI_\phi/\alpha^3a^3)(-c_{17}U_{rs} - c_{18}a\alpha\Theta_{rs}' + \\
&\qquad c_{15}U_{r+1,\,s} - c_{16}a\alpha\Theta_{r+1,\,s}') \\
-M_{r+1,\,s}{}'^L &= (EI_\phi/\alpha^2a^2)(c_{18}U_{rs} + c_{20}a\alpha\Theta_{rs}' - \\
&\qquad c_{16}U_{r+1,\,s} + c_{19}a\alpha\Theta_{r+1,\,s}')
\end{aligned}\right\} \quad (5)$$

where

$$\left.\begin{aligned}
c_{15} &= p^3(\cosh p \sin p + \sinh p \cos p)/(1 - \cos p \cosh p) \\
c_{16} &= p^2[\sin p \sinh p/(1 - \cos p \cosh p)] - [GJ\alpha^2/EI_\phi] \\
c_{17} &= p^3(\sin p + \sinh p)/(1 - \cos p \cosh p) \\
c_{18} &= p^2(\cosh p - \cos p)/(1 - \cos p \cosh p) \\
c_{19} &= p(\cosh p \sin p - \cos p \sinh p)/(1 - \cos p \cosh p) \\
c_{20} &= p(\sinh p - \sin p)/(1 - \cos p \cosh p) \\
p &= (\omega^2A\rho\alpha^4a^4/EI_\phi)^{1/4}
\end{aligned}\right\} \quad (6)$$

These flexural vibrations induce torsional moments at each end. In the present formulation, however, they cancel identically at each node and do not enter the equations of equilibrium. Note that the shearing force Q_{rs}' is in addition to the shearing forces induced by torsional vibrations. Θ_{rs}' is the slope of the ring at (r, s).

Turning next to the flexural vibrations of the stringers, the equations relating the forces and displacements may be written as in Ref. 1 for flexural motion in the radial direction (Fig. 4). The first of these equations is

$$-\bar{Q}_{rs}{}^{R} = (E\bar{I}/b^3)(\bar{c}_{15}W_{rs} + \bar{C}_{16}b\bar{\Theta}_{rs} - \bar{C}_{17}W_{r,\, s+1} + \bar{c}_{18}b\bar{\Theta}_{r,\, s+1}) \quad (7)$$

The other equations may be written down by analogy with (5). In Eq. (7), $\bar{c}_{15}$, $\bar{c}_{16}$, etc., are the same functions of $\bar{p}$ as c_{15}, c_{16}, etc., are of p where

$$\bar{p} = (\omega^2\bar{A}\rho b^4/E\bar{I})^{1/4} \quad (8)$$

in which $\bar{A}$ is the cross-sectional area of a stringer, $\bar{I}$ its moment of inertia, and ρ its mass density ($\bar{c}_{16}$ does not include the GJ term involved in c_{16}).

Equations similar to the foregoing apply for the case of flexure in the tangential direction. The rotations in this case are denoted by Θ', and the sign convention used is the same as in Fig. 4. The first of these equations is

$$-\bar{Q}_{rs}{}'^{R} = (E\bar{I}_{\phi}/b^3)(c_{15}'V_{rs} + c_{16}'a\Theta_{rs}' - c_{17}'V_{r,\, s+1} + c_{18}'b\Theta_{r,\, s+1}') \quad (9)$$

where c_{15}', c_{16}', . . . , c_{20}' are the same functions of

$$p' = (\omega^2\bar{A}\rho b^4/E\bar{I}_{\phi})^{1/4}$$

as $\bar{c}_{15}$, $\bar{c}_{16}$, . . . , $\bar{c}_{20}$ are of $\bar{p}$.

Considering next the longitudinal motion of the stringers, it may be shown that the relation between the forces and displacements (Fig. 4) is given by

$$\left.\begin{aligned} \bar{N}_{rs}{}^{R} &= (\bar{A}E/b)(-c_{21}U_{rs} + c_{22}U_{r,\, s+1}) \\ \bar{N}_{r,\, s+1}{}^{L} &= (\bar{A}E/b)(c_{21}U_{r,\, s+1} - c_{22}U_{rs}) \end{aligned}\right\} \quad (10)$$

with

$$c_{21} = j\cot j \qquad c_{22} = j\operatorname{cosec} j \qquad j = (\omega^2\rho b^2/E)^{1/2} \quad (11)$$

U_{rs} is the amplitude of the longitudinal displacement at (r, s) considered positive in the positive direction of s.

The torsional vibrations of the stringer may be investigated as in Ref. 1. One finds

$$T_{rs} = -(G\bar{J}/b)[c_{24}(\Theta_{r,\, s+1} + \Theta_{r,\, s-1}) - 2c_{23}\Theta_{rs}] \quad (12)$$

with

$$c_{23} = l\cot l \qquad c_{24} = l\operatorname{cosec} l \qquad l = (b^2\rho\bar{I}_p\omega^2/G\bar{J})^{1/2} \quad (13)$$

where $\bar{J}$ and $\bar{I}_p$ are the torsion constant and polar moment of inertia, respectively, of the stringer cross section.

Equilibrium of a "joint" requires that

$$\left.\begin{aligned} M_{rs}{}^{R} - M_{rs}{}^{L} + T_{rs} &= 0 \\ \bar{M}_{rs}{}^{R} - \bar{M}_{rs}{}^{L} + \bar{T}_{rs} &= 0 \\ Q_{rs}{}^{L} - Q_{rs}{}^{R} + \bar{Q}_{rs}{}^{L} - \bar{Q}_{rs}{}^{R} &= 0 \\ N_{rs}{}^{L} - N_{rs}{}^{R} + \bar{Q}_{rs}{}'^{L} - \bar{Q}_{rs}{}'^{R} &= 0 \\ \bar{N}_{rs}{}^{R} - \bar{N}_{rs}{}^{L} + Q_{rs}{}'^{R} - Q_{rs}{}'^{L} - Q_{rs}' &= 0 \\ M_{rs}{}'^{R} - \bar{M}_{rs}{}'^{R} - M_{rs}{}'^{L} + \bar{M}_{rs}{}'^{L} &= 0 \end{aligned}\right\} \quad (14)$$

Introducing the shifting operator defined by

$$\mathbf{E}_r{}^{x}f_{rs} = f_{r+x,\, s} \qquad \mathbf{E}_s{}^{x}f_{rs} = f_{r,\, s+x}$$

using Eqs. (1–13), and eliminating Θ_{rs}, $\bar{\Theta}_{rs}$, and Θ_{rs}' from the resulting equations, we get the following three equilibrium equations containing only the displacement components, U_{rs},

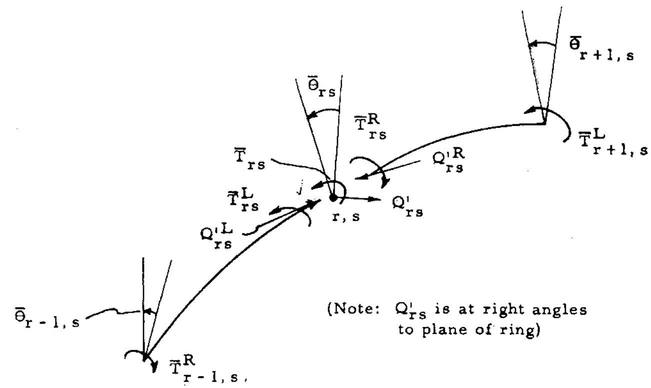

Fig. 3 Torsion of ring segment.

V_{rs}, and W_{rs}:

$$\left[\frac{c_{10}{}^2(\mathbf{E}_r - \mathbf{E}_r{}^{-1})^2}{\xi_{rs}} + c_9(\mathbf{E}_r + \mathbf{E}_r{}^{-1}) + 2c_7 + \lambda\left\{\frac{\bar{c}_{18}{}^2(\mathbf{E}_s - \mathbf{E}_s{}^{-1})^2}{\mathbf{n}_{rs}} - \bar{c}_{17}(\mathbf{E}_s + \mathbf{E}_s{}^{-1}) + 2\,\bar{c}_{15}\right\}\right]W_{rs} - \left\{c_5 + \frac{c_6c_{10}(\mathbf{E}_r + \mathbf{E}_r{}^{-1}) + 2c_3c_{10}}{\xi_{rs}}\right\}(\mathbf{E}_r - \mathbf{E}_r{}^{-1})V_{rs} = 0 \quad (15)$$

$$\left\{c_5 + \frac{c_6c_{10}(\mathbf{E}_r + \mathbf{E}_r{}^{-1}) + 2c_3c_{10}}{\xi_{rs}}\right\}(\mathbf{E}_r - \mathbf{E}_r{}^{-1})W_{rs} + \left[2c_1 + c_4(\mathbf{E}_r + \mathbf{E}_r{}^{-1}) - \frac{\{c_6(\mathbf{E}_r + \mathbf{E}_r{}^{-1}) + 2c_3\}^2}{\xi_{rs}} + \lambda_1\{2c_{15}' - c_{17}'(\mathbf{E}_s + \mathbf{E}_s{}^{-1})\} - \frac{\lambda_1\lambda_2 b}{a\alpha}\,\frac{c_{18}'^2(\mathbf{E}_s - \mathbf{E}_s{}^{-1})^2}{\zeta_{rs}}\right]V_{rs} + \frac{\lambda_1 b}{a\alpha}\left\{\frac{c_{18}c_{18}'(\mathbf{E}_s - \mathbf{E}_s{}^{-1})(\mathbf{E}_r - \mathbf{E}_r{}^{-1})}{\zeta_{rs}}\right\}U_{rs} = 0 \quad (16)$$

$$\mu_1\bar{C}_{18}(\mathbf{E}_s - \mathbf{E}_s{}^{-1})\frac{\{c_{14}(\mathbf{E}_r + \mathbf{E}_r{}^{-1}) - 2c_{13}\}}{\mathbf{n}_{rs}}W_{rs} + \mu\lambda_2c_{18}c_{18}'\frac{(\mathbf{E}_r - \mathbf{E}_r{}^{-1})(\mathbf{E}_s - \mathbf{E}_s{}^{-1})}{\zeta_{rs}}V_{rs} + \left[c_{22}(\mathbf{E}_s + \mathbf{E}_s{}^{-1}) - 2c_{21} - \mu\left\{2c_{15} - c_{17}(\mathbf{E}_r + \mathbf{E}_r{}^{-1}) + \frac{c_{18}{}^2(\mathbf{E}_r - \mathbf{E}_r{}^{-1})^2}{\zeta_{rs}}\right\}\right]U_{rs} = 0 \quad (17)$$

where

$$\left.\begin{aligned} \xi_{rs} &= c_{12}(\mathbf{E}_r + \mathbf{E}_r{}^{-1}) - \nu c_{24}(\mathbf{E}_s + \mathbf{E}_s{}^{-1}) + 2(c_{11} + \nu c_{23}) \\ \mathbf{n}_{rs} &= \bar{c}_{20}(\mathbf{E}_s + \mathbf{E}_s{}^{-1}) - \nu_1c_{14}(\mathbf{E}_r + \mathbf{E}_r{}^{-1}) + 2(\bar{c}_{19} + \nu_1c_{13}) \\ \zeta_{rs} &= 2c_{19} + c_{20}(\mathbf{E}_r + \mathbf{E}_r{}^{-1}) - (\lambda_2b/a\alpha)\times[2c_{19}' + c_{20}'(\mathbf{E}_s + \mathbf{E}_s{}^{-1})] \\ \lambda &= \frac{\bar{I}a^3\alpha^3}{Ib^3} \qquad \lambda_1 = \frac{\bar{I}_{\phi}a^3\alpha^3}{Ib^3} \qquad \lambda_2 = \frac{\bar{I}_{\phi}a^2\alpha^2}{I_{\phi}b^2} \\ \nu &= \left(\frac{G\bar{J}}{b}\right)\left(\frac{a\alpha}{EI}\right) \qquad \nu_1 = \left(\frac{GJ}{a\alpha}\right)\left(\frac{b}{E\bar{I}}\right) \\ \mu &= \frac{I_{\phi}b}{a^3\alpha^3\bar{A}} \qquad \mu_1 = \frac{\epsilon GJ}{E\alpha a^2\bar{A}} \qquad \epsilon = \left(1 + \frac{EI_{\phi}}{GJ}\right) \end{aligned}\right\} \quad (18)$$

The procedure for finding the frequency equation is to assume a set of displacement functions W_{rs}, V_{rs}, U_{rs} that satisfies given boundary conditions. Introduction of these functions

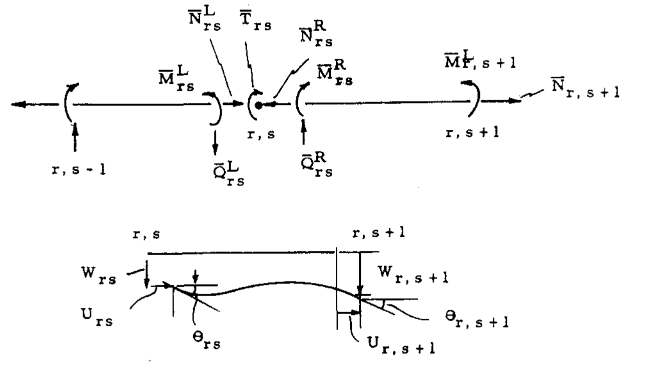

Fig. 4 Forces and displacements of stringer (flexural motion in radial direction and longitudinal motion).

into Eqs. (15–17) gives three homogeneous equations between the amplitudes of these displacements. The condition that the determinant of these coefficients vanish yields the frequency equation sought.

Simply Supported Shell

Assume that the shell is simply supported at $s = 0, S$. A suitable set of boundary conditions is as follows:

$$\left.\begin{aligned} &W_{rs} = 0 \qquad V_{rs} = 0 \qquad \Theta_{rs} = 0 \\ &\bar{M}_{rs}{}^L - (\bar{T}_{rs}/2) = \bar{M}_{rs}{}^R + (\bar{T}_{rs}/2) = 0 \\ &\bar{N}_{rs}{}^L + \frac{Q_{rs}'}{2} + \frac{Q_{rs}'^L}{2} - \frac{Q_{rs}'^R}{2} = \\ &\qquad \bar{N}_{rs}{}^R - \frac{Q_{rs}'}{2} + \frac{Q_{rs}'^R}{2} - \frac{Q_{rs}'^L}{2} = 0 \\ &\bar{M}_{rs}'^R - (M_{rs}'^R - M_{rs}'^L)/2 = \\ &\qquad \bar{M}_{rs}'^L + M_{rs}'^R - M_{rs}'^L/2 = 0 \end{aligned}\right\} \text{at } s = 0, S$$

Of the six boundary conditions for the stringer, two relate to the flexural vibration in the radial direction and two to the flexural vibration in the tangential direction, since each of these motions is governed by a fourth-order differential equation. The torsional vibration and the axial vibrations are each governed by second-order differential equations, so that each requires one additional boundary condition, giving a total of six at each edge.

The first three conditions are self-explanatory. The fourth and sixth conditions require the vanishing of the bending moments in the stringers in two perpendicular directions, and the fifth condition requires the vanishing of the normal force in the stringer. However, as stated here, the last three conditions assume that at the boundary of the shell there exists a "half-ring" similar to the other rings of the system. It is for this reason that the fourth boundary condition, e.g., demands that $\bar{M}_{rs}{}^R + (\bar{T}_{rs}/2) = 0$ instead of simply $\bar{M}_{rs}{}^R = 0$ as would be the case if the half-ring were absent. This slight modification in the boundary condition is necessitated by the difficulty of terminating the mathematical model to the right or left of a ring instead of along its centerline. The effect of these half-rings may be expected to decrease rapidly as the number of bays increases. Furthermore, they render the formulation of the solution considerably simpler as will be seen below.

The preceding boundary conditions are equivalent to the following six conditions:

$$\left.\begin{aligned} &W_{rs} = 0 \qquad V_{rs} = 0 \qquad \Theta_{rs} = 0 \\ &\bar{M}_{rs}{}^R + \bar{M}_{rs}{}^L = 0 \\ &\bar{N}_{rs}{}^L + \bar{N}_{rs}{}^R = 0 \\ &\bar{M}_{rs}'^R + \bar{M}_{rs}'^L = 0 \end{aligned}\right\} \text{at } s = 0, S \tag{19}$$

Using the formulas derived previously, these conditions may be written in terms of the displacement components W_{rs}, V_{rs}, and U_{rs} as follows:

$$\left.\begin{aligned} &W_{rs} = 0 \qquad V_{rs} = 0 \\ &\Theta_{rs} = c_{10}\frac{(\mathbf{E}_r - \mathbf{E}_r^{-1})}{\xi_{rs}}\frac{W_{rs}}{a\alpha} - \left\{\frac{c_6(\mathbf{E}_r + \mathbf{E}_r^{-1}) + 2c_3}{\xi_{rs}}\right\}\frac{V_{rs}}{a\alpha} = 0 \\ &\{2\bar{c}_{16} - \bar{c}_{18}(\mathbf{E}_s + \mathbf{E}_s^{-1}) + \bar{c}_{18}\bar{c}_{20} \times [(\mathbf{E}_s - \mathbf{E}_s^{-1})^2/\mathbf{n}_{rs}]\}\, W_{rs} = 0 \\ &(\mathbf{E}_s - \mathbf{E}_s^{-1})U_{rs} = 0 \\ &\{2c_{16}' - c_{18}'(\mathbf{E}_s + \mathbf{E}_s^{-1}) + c_{18}'c_{20}'\lambda_2(b/a\alpha) \times [(\mathbf{E}_s - \mathbf{E}_s^{-1})^2/\zeta_{rs}]V_{rs} + c_{20}'c_{19}(b/a\alpha)(\mathbf{E}_r - \mathbf{E}_r^{-1}) \times [(\mathbf{E}_s - \mathbf{E}_s^{-1})/\zeta_{rs}]U_{rs} = 0 \end{aligned}\right\} \tag{20}$$

Consider now a set of displacement functions:

$$\left.\begin{aligned} W_{rs} &= A\exp(i2n\pi r/R)\sin(m\pi s/S) \\ V_{rs} &= B\exp(i2n\pi r/R)\sin(m\pi s/S) \\ n &= 2, 3, \ldots, n < R/2 \\ m &= 1, 2, \ldots, S - 1 \\ U_{rs} &= C\exp(i2n\pi/R)\cos(m\pi s/S) \end{aligned}\right\} \tag{21}‡$$

in which A, B, and C are arbitrary constants, and n and m are integers. These functions obviously satisfy the first two boundary conditions (20) at $s = 0, S$.

It is now noted that, for all the operators occurring here,

$$F(\mathbf{E}_r, \mathbf{E}_s)a^r b^s = a^r b^s F(a, b) \tag{22}$$

Introducing, e.g., the first of (21) into the fourth of (20)

$$\text{L.H.S.} = A\exp\frac{i2n\pi r}{R}\left\{2\bar{c}_{16} - \bar{c}_{18}(\mathbf{E}_s + \mathbf{E}_s^{-1}) + \frac{\bar{c}_{18}\bar{c}_{20}(\mathbf{E}_s - \mathbf{E}_s^{-1})^2}{\bar{c}_{20}(\mathbf{E}_s + \mathbf{E}_s^{-1}) - 2\nu_1 c_{14}\cos(2n\pi/R) + 2(\bar{c}_{19} + \nu_1 c_{13})}\right\}\cdot$$

$$Im\exp\left(\frac{im\pi s}{S}\right) = A\exp\frac{i2n\pi r}{R}\, Im\left\{2\bar{c}_{16} - 2\bar{c}_{18}\cos\frac{m\pi}{S} - \frac{4\bar{c}_{18}\bar{c}_{20}\sin^2(m\pi/S)}{2\bar{c}_{20}\cos(m\pi/S) - 2\nu_1 c_{14}\cos(2n\pi/R) + 2(\bar{c}_{19} + \nu_1 c_{13})}\right\}\cdot$$

$$\exp\left(\frac{im\pi s}{S}\right) = A\left\{2\bar{c}_{16} - 2\bar{c}_{18}\cos\frac{m\pi}{S} - \frac{4\bar{c}_{18}\bar{c}_{20}\sin^2(m\pi/S)}{2\bar{c}_{20}\cos(m\pi/S) - 2\nu_1 c_{14}\cos(2n\pi/R) + 2(\bar{c}_{19} + \nu_1 c_{13})}\right\}\cdot$$

$$\exp\frac{2in\pi r}{R}\sin\frac{m\pi s}{S}$$

which vanishes at $s = 0, S$, thus satisfying the fourth of the boundary conditions (20). Similarly, it may be shown that the displacement functions (21) satisfy the remaining boundary conditions (20).

Introducing the functions (21) into Eqs. (15–17) yields the three homogeneous equations

$$\begin{gathered} A\gamma_1 - iB\gamma_2 = 0 \qquad -iA\gamma_2 - B\gamma_3 + iC\gamma_4 = 0 \\ A\gamma_5 + iB\gamma_4 + C\gamma_6 = 0 \end{gathered} \tag{23}$$

‡ The reason for the upper limits on n and m is discussed in the following section.

where

$$\gamma_1 = c_7 + c_9 \cos\frac{2n\pi}{R} - \frac{c_{10}^2 \sin^2(2n\pi/R)}{c_{12}\cos(2n\pi/R) - \nu c_{24}\cos(m\pi/S) + c_{11} + \nu c_{23}} + \lambda\left\{\bar{c}_{15} - \bar{c}_{17}\cos\frac{m\pi}{S} - \frac{\bar{c}_{18}^2 \sin^2(m\pi/S)}{\bar{c}_{20}\cos(m\pi/S) - \nu_1 c_{14}\cos(2n\pi/R) + \bar{c}_{19} + \nu_1 c_{23}}\right\}$$

$$\gamma_2 = \left\{c_5 + \frac{c_{10}[c_3 + c_6\cos(2n\pi/R)]}{c_{12}\cos(2n\pi/R) - \nu c_{24}\cos(m\pi/S) + c_{11} + \nu c_{23}}\right\}\sin\frac{2n\pi}{R}$$

$$\gamma_3 = c_1 + c_4\cos\frac{2n\pi}{R} - \frac{[c_3 + c_6\cos(2n\pi/R)]^2}{c_{12}\cos(2n\pi/R) - \nu c_{24}\cos(m\pi/S) + c_{11} + \nu c_{23}} + \lambda_1\left(c_{15}' - c_{17}'\cos\frac{m\pi}{S}\right) + \frac{\lambda_1\lambda_2 b}{a\alpha}\cdot\frac{c_{18}'^2\sin^2(m\pi/S)}{c_{19} + c_{20}\cos(2n\pi/R) - (\lambda_2 b/a\alpha)[c_{19}' + c_{20}'\cos(m\pi/S)]} \tag{24}$$

$$\gamma_4 = \frac{\lambda_1 b}{a\alpha}\,\frac{c_{18}c_{18}'\cos(m\pi/S)\sin(2n\pi/R)}{c_{19} + c_{20}\cos(2n\pi/R) - (\lambda_2 b/a\alpha)[c_{19}' + c_{20}'\cos(m\pi/S)]}$$

$$\gamma_5 = \bar{c}_{18}\,\frac{[c_{14}\cos(2n\pi/R) - c_{13}]\sin(m\pi/S)}{\bar{c}_{20}\cos(m\pi/S) - \nu_1 c_{14}\cos(2n\pi/R) + \bar{c}_{19} + \nu_1 c_{13}} \times \frac{\lambda_1 b\mu_1}{\mu\lambda_2 a\alpha}$$

$$\gamma_6 = \left[c_{22}\cos\frac{m\pi}{S} - c_{21} - \mu\left\{c_{15} - c_{17}\cos\frac{2n\pi}{R} - \frac{c_{18}^2\sin^2(2n\pi/R)}{c_{19} + c_{20}\cos(2n\pi/R) - (\lambda_2 b/a\alpha)[c_{19}' + c_{20}'\cos(m\pi/S)]}\right\}\right] \times \frac{\lambda_1 b}{\mu\lambda_2 a\alpha}$$

The condition that the homogeneous system (23) yields nontrivial solutions gives the frequency equation

$$\gamma_6(\gamma_1\gamma_3 - \gamma_2^2) - \gamma_4(\gamma_1\gamma_4 + \gamma_2\gamma_5) = 0 \tag{25}$$

This transcendental equation has to be solved by trial for the frequencies ω. For each value of m and n chosen, the equation yields an infinity of frequencies. It is noted that the first of Eqs. (23) lacks a third term for complete symmetry of the matrix of the system. This arises from using various approximations in the differential equations for torsional and out-of-plane ring vibrations that, strictly speaking, cannot be uncoupled. In view of the already stated limitations of the model for predominantly u-displacement modes, greater complexity does not appear warranted.

Lumped Mass Approximation

The frequency equation (25) is a transcendental equation of some complexity, and it is desirable to have an approximate solution if feasible. It is intuitively evident that a suitable approximation may be obtained by lumping the masses of the rings and stringers at the nodes, thereby reducing the system to one of a finite number of degrees of freedom. If this is done it may be shown (see Appendix) that Eqs. (1) still hold but the coefficients c_i may be written down explicitly as follows:

$$\left.\begin{aligned}
c_1 &= 2\alpha^3(\alpha\sin\alpha\cos\alpha - \alpha^2 + 2\cos^2\alpha - 4\cos\alpha + 2)/D \\
c_2 &= 2\alpha^3(\alpha\sin^2\alpha + 2\sin\alpha\cos\alpha - 2\sin\alpha)/D \\
c_3 &= 2\alpha^2(\alpha^2 - \alpha\sin\alpha - \alpha\sin\alpha\cos\alpha - 3\cos^2\alpha + 4\cos\alpha - 1)/D \\
c_4 &= 2\alpha^3(\alpha^2\cos\alpha - \alpha\sin\alpha - 4\cos\alpha + 2\cos^2\alpha + 2)/D \\
c_5 &= 2\alpha^3\sin\alpha(2 - \alpha^2 - 2\cos\alpha)/D \\
c_6 &= 2\alpha^2(2\alpha\sin\alpha - \alpha^2\cos\alpha - \cos^2\alpha + 4\cos\alpha - 3)/D \\
c_7 &= 2\alpha^3(2\sin^2\alpha - \alpha^2 - \alpha\sin\alpha\cos\alpha)/D \\
c_8 &= 2\alpha^2(3\sin\alpha + \alpha\cos\alpha - \alpha - \alpha\sin^2\alpha - 3\sin\alpha\cos\alpha)/D \\
c_9 &= 2\alpha^3(\alpha^2\cos\alpha + \alpha\sin\alpha - 2\sin^2\alpha)/D \\
c_{10} &= 2\alpha^2(\alpha - \alpha^2\sin\alpha - \alpha\cos\alpha + \sin\alpha - \sin\alpha\cos\alpha)/D \\
c_{11} &= \alpha(4\alpha\sin\alpha - 3\alpha^2 + 2\alpha\sin\alpha\cos\alpha + 7\cos^2\alpha - 8\cos\alpha + 1)/D \\
c_{12} &= \alpha(2\alpha^2\cos\alpha - 6\alpha\sin\alpha - 8\cos\alpha + \cos^2\alpha + \alpha^2 + 7)/D \\
D &= \alpha\sin^2\alpha + 4\alpha - \alpha^3 - 4\alpha\cos\alpha + 4\sin\alpha\cos\alpha - 4\sin\alpha
\end{aligned}\right\} \tag{26}$$

The c_i now do not contain the unknown frequency ω and depend only on the subtended angle α.

Equations (4–13) also simplify with the various coefficients c_i being given as follows:

$$\left.\begin{aligned}
&c_{13} = q\coth q \qquad c_{14} = q\,\mathrm{csch}\,q \\
&\qquad\qquad q = \alpha(EI_\phi/GJ)^{1/2} \\
&c_{15} = \bar{c}_{15} = c_{15}' = c_{17} = \bar{c}_{17} = c_{17}' = 12 \\
&\bar{c}_{16} = c_{16}' = c_{18} = \bar{c}_{18} = c_{18}' = 6 \\
&c_{16} = 6 - (GJ\alpha^2/EI_\phi) \\
&c_{19} = \bar{c}_{19} = c_{19}' = 4 \qquad c_{20} = \bar{c}_{20} = c_{20}' = 2
\end{aligned}\right\} \tag{27}$$

$$c_{21} = c_{22} = c_{23} = c_{24} = 1$$

Denoting by ρ^* the mass lumped at each intersection, one can now proceed as before and obtain the following homogeneous equations:

$$\left.\begin{aligned}
A[\bar{\gamma}_1 - (\rho^*\omega^2 a^3\alpha^3/2EI)] - iB\bar{\gamma}_2 &= 0 \\
iA\bar{\gamma}_2 + B[\bar{\gamma}_3 - (\rho^*\omega^2 a^3\alpha^3/2EI)] - iC\bar{\gamma}_4 &= 0 \\
A\bar{\gamma}_5 + iB\bar{\gamma}_4 + C[\bar{\gamma}_6 + (\rho^*\omega^2 a^3\alpha^3/2EI)] &= 0
\end{aligned}\right\} \tag{28}$$

where $\bar{\gamma}_1, \bar{\gamma}_2 \ldots$ may be obtained from Eqs. (24) by using the relations (27).

The requirement that the homogeneous system (28) yield nontrivial solutions leads to the (cubic) frequency equation

$$\Omega^3 + \bar{A}\Omega^2 + \bar{B}\Omega + \bar{C} = 0 \tag{29}$$

where

$$\left.\begin{aligned}
\Omega &= \rho^*\omega^2 a^3\alpha^3/EI \qquad A = 2(\bar{\gamma}_6 - \bar{\gamma}_1 - \bar{\gamma}_3) \\
\bar{B} &= 4(\bar{\gamma}_1\bar{\gamma}_3 - \bar{\gamma}_3\bar{\gamma}_6 - \bar{\gamma}_1\bar{\gamma}_6 - \bar{\gamma}_2^2 + \bar{\gamma}_4^2) \\
\bar{C} &= 8(\bar{\gamma}_1\bar{\gamma}_3\bar{\gamma}_6 - \bar{\gamma}_2^2\bar{\gamma}_6 - \bar{\gamma}_1\bar{\gamma}_4^2 - \bar{\gamma}_2\bar{\gamma}_5\bar{\gamma}_4)
\end{aligned}\right\} \tag{30}$$

In contrast to (25), Eq. (29) may be solved explicitly. However, for chosen m and n, Eq. (29) yields only three frequencies ω.

Numerical Example

In order to crystallize ideas, an arbitrary example of a gridwork shell with rings made up of $10 \times 4\frac{5}{8} \times 25.4\#$ American standard beams and stringers of $15 \times 5\frac{1}{2} \times 42.9\#$ American standard beams was assumed. The number of bays S was taken as 6 and R, the number of stringers as 8. The results of the computation are shown in Table 1.

The table lists three values of the frequency parameter $\bar{p}$ obtained from the distributed mass formulation [Eq. (25)] and the three roots of the cubic equation (29) which represent the lumped mass approximation.

It is seen that there is reasonable agreement between the two solutions only for the lowest natural frequencies tabulated as $(\bar{p})_1$. The higher roots involving larger v and u displacements display no pattern of agreement. It is remarked in this regard that the transcendental equation (25) is quite difficult to solve even on an electronic computer because of numerous discontinuities in the functions involved, and some error in the values of the higher roots is possible. The basic difficulty appears to be errors introduced in calculating the coefficients c_i in Eqs. (1), which involves the inversion of a 6×6 matrix followed by a matrix multiplication. Some of the difficulty could thus be eliminated if the c_i could be written down explicitly as in the lumped mass formulation. This, however, turns out to be a formidable task.

Equation (29), of course, is very easily solved and seems, therefore, suitable for the calculation of the lowest frequencies of a gridwork shell. However, a more extensive parametric study would be required to decide for what range of grid spacing the lumped mass formulation is a good approximation.

We remark that, in Eqs. (21), and therefore in Eqs. (23) and (28), n and m have been limited by maximum values $<R/2$ and $S - 1$, respectively. For $n > R/2$ and $m > S$, it is easily verified that the solutions will merely repeat themselves. On the other hand, when $n = R/2$ and $m = S$, Eqs. (23) and (28) cannot be deduced. This is a consequence of the fact that, when Eqs. (21) are introduced into the equilibrium equations [e.g., (15–17)], one or another of the resulting equations will always have variable coefficients of the form $\sin(2n\pi r/R)$ or $\sin(m\pi s/S)$, and Eqs. (23) and (28) follow only if these variable coefficients do not vanish for all r or all s. Obviously this condition no longer holds when $2n = R$ or $m = S$ and indeed whenever $2n$ is an integral multiple of R, or m is an integral multiple of S. The present method is entirely applicable to such cases, but such special forms of vibration where one or more of the nodal displacements vanish identically have to be formulated separately starting from the equilibrium equations.

Appendix

On the basis of the assumptions given under Introduction, the equations of motion given by Love[2] may be generalized as follows.

Flexural Motion of Ring

In the notation of Fig. 1, the motion is governed by

$$\frac{EI}{a^4}\left(\frac{\partial^6 v}{\partial\phi^6} + 2\frac{\partial^4 v}{\partial\phi^4} + \frac{\partial^2 v}{\partial\phi^2}\right) - \rho A \frac{\partial^2}{\partial t^2}\left(v - \frac{\partial^2 v}{\partial\phi^2}\right) = 0 \quad \text{(A1)}$$

together with the condition of inextensibility $\partial v/\partial\phi = w$. In Eq. (A1), EI is the flexural rigidity about a centroidal axis of the ring cross section at right angles to the plane of the ring of radius a, v is the tangential displacement, w the inward radial displacement, A the cross-sectional area, and ρ the volume density.

Assuming

$$v = V(\phi)\exp i\omega t \quad \text{(A2)}$$

where ω is the natural circular frequency, substitution into (A1) leads to

$$\frac{d^6V}{d\phi^6} + 2\frac{d^4V}{d\phi^4} + \frac{d^2V}{d\phi^2}(1 - K^2) + K^2V = 0 \quad \text{(A3)}$$

where $K^2 = \rho A\omega^2 a^4/EI$. The characteristic equation for (A3) is

$$\Lambda^6 + 2\Lambda^4 + \Lambda^2(1 - K^2) + K^2 = 0$$

Considering this as a cubic in Λ^2, it may be shown that Z^2 has one negative value and that the remaining two roots are both real and positive provided that

$$K^2 > 17.6366 \quad \text{(A4)}$$

If the inequality (A4) holds, the general solution valid for the circular segment with central angle α is of the form

$$V = B_1 \operatorname{ch}(\beta_1/\alpha)\phi + B_2 \operatorname{sh}(\beta_1/\alpha)\phi + B_3 \operatorname{ch}(\beta_2/\alpha)\phi + B_4 \operatorname{sh}(\beta_2/\alpha)\phi + B_5 \cos(\beta_3/\alpha)\phi + B_6 \sin(\beta_3/\alpha)\phi \quad \text{(A5)}$$

in which we have written ch for cosh and sh for sinh, and β_1^2/α^2, β_2^2/α^2, $-\beta_3^2/\alpha^2$ are the three values of Λ^2, the B's being arbitrary.

If one uses the boundary conditions

$$V = V_{rs} \qquad W = W_{rs} \qquad 1/a[V + (dW/d\phi)] = \Theta_{rs} \quad \text{at } \phi = 0 \quad \text{(A6)}$$

$$V = V_{r+1,\,s} \qquad W = W_{r+1,\,s} \qquad 1/a[V + (dW/d\phi)] = \Theta_{r+1,\,s} \quad \text{at } \phi = \alpha \quad \text{(A7)}$$

there results the matrix equation

$$[g_{ij}]\{B_j\} = \{k_i\} \quad \text{(A8)}$$

where

$$[g_{ij}] \equiv \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & \frac{\beta_1}{\alpha} & 0 & \frac{\beta_2}{\alpha} & 0 & \frac{\beta_3}{\alpha} \\ \left(1 + \frac{\beta_1^2}{\alpha^2}\right)\alpha & 0 & \left(1 + \frac{\beta_2^2}{\alpha^2}\right)\alpha & 0 & \left(1 - \frac{\beta_3^2}{\alpha^2}\right)\alpha & 0 \\ \operatorname{ch}\beta_1 & \operatorname{sh}\beta_1 & \operatorname{ch}\beta_2 & \operatorname{sh}\beta_2 & \cos\beta_3 & \sin\beta_3 \\ \frac{\beta_1}{\alpha}\operatorname{sh}\beta_1 & \frac{\beta_1}{\alpha}\operatorname{ch}\beta_1 & \frac{\beta_2}{\alpha}\operatorname{sh}\beta_2 & \frac{\beta_2}{\alpha}\operatorname{ch}\beta_2 & -\frac{\beta_3}{\alpha}\sin\beta_3 & \frac{\beta_3}{\alpha}\cos\beta_3 \\ \alpha\left(1 + \frac{\beta_1^2}{\alpha^2}\right)\operatorname{ch}\beta_1 & \alpha\left(1 + \frac{\beta_1^2}{\alpha^2}\right)\operatorname{sh}\beta_1 & \alpha\left(1 + \frac{\beta_2^2}{\alpha^2}\right)\operatorname{ch}\beta_2 & \alpha\left(1 + \frac{\beta_2^2}{\alpha^2}\right)\operatorname{sh}\beta_2 & \alpha\left(1 - \frac{\beta_3^2}{\alpha^2}\right)\operatorname{sh}\beta_3 & \alpha\left(1 - \frac{\beta_3^2}{\alpha^2}\right)\sin\beta_3 \end{bmatrix} \quad \text{(A9)}$$

$$\{k_i\} = \begin{Bmatrix} V_{rs} \\ W_{rs} \\ \alpha a\Theta_{rs} \\ V_{r+1,\,s} \\ W_{r+1,\,s} \\ \alpha a\Theta_{r+1,\,s} \end{Bmatrix} \tag{A10}$$

From Fig. 2, one has

$$-\frac{EI}{a^2}\left(W + \frac{d^2W}{d\phi^2}\right) = M_{rs}{}^R \text{ at } \phi = 0$$

$$= M_{r+1,\,s}{}^L \text{ at } \phi = \alpha$$

$$-\frac{EI}{a^3}\left(\frac{dW}{d\phi} + \frac{d^3W}{d\phi^3}\right) = Q_{rs}{}^R \text{ at } \phi = 0 \tag{A11}$$

$$= Q_{r+1,\,s}{}^L \text{ at } \phi = \alpha$$

$$\frac{EI}{a^3}\left(\frac{d^4W}{d\phi^4} + \frac{d^2W}{d\phi^2} - K^2W\right) = N_{rs}{}^R \text{ at } \phi = 0$$

$$= N_{r+1,\,s}{}^L \text{ at } \phi = \alpha$$

Substitution of (A5) [with $W = (dV/d\phi)$] into (A11) leads to the matrix equation

$$[h_{ij}]\{B_j\} = \{l_i\} \tag{A12}$$

where

$$[h_{ij}] \equiv \begin{bmatrix} 0 & -\xi_1 & 0 & -\xi_2 & 0 & \xi_3 \\ \eta_1 & 0 & \eta_2 & 0 & \eta_3 & 0 \\ 0 & -\zeta_1 & 0 & -\zeta_2 & 0 & -\zeta_3 \\ \xi_1\,\mathrm{sh}\beta_1 & \xi_1\,\mathrm{ch}\beta_1 & \xi_2\,\mathrm{sh}\beta_2 & \xi_2\,\mathrm{ch}\beta_2 & \xi_3\sin\beta_3 & -\xi_3\cos\beta_3 \\ -\eta_1\,\mathrm{ch}\beta_1 & -\eta_1\,\mathrm{sh}\beta_1 & -\eta_2\,\mathrm{ch}\beta_2 & -\eta_2\,\mathrm{sh}\beta_2 & -\eta_3\cos\beta_3 & -\eta_3\sin\beta_3 \\ \zeta_1\,\mathrm{sh}\beta_1 & \zeta_1\,\mathrm{ch}\beta_1 & \zeta_2\,\mathrm{sh}\beta_2 & -\zeta_2\,\mathrm{ch}\beta_2 & -\zeta_3\sin\beta_3 & \zeta_3\cos\beta_3 \end{bmatrix} \tag{A13}$$

$$\{l_i\} \equiv \begin{Bmatrix} -N_{rs}{}^R a\alpha \\ -Q_{rs}{}^R a\alpha \\ M_{rs}{}^R \\ N_{r+1,\,s}{}^L a\alpha \\ Q_{r+1,\,s}{}^L a\alpha \\ -M_{r+1,\,s}{}^L \end{Bmatrix} \frac{a^2\alpha^2}{EI} \tag{A14}$$

$$\xi_1 = \alpha^2\beta_1\left(\frac{\beta_1{}^4}{\alpha^4} + \frac{\beta_1{}^2}{\alpha^2} - K^2\right) = -\frac{2K^2\beta_1\alpha^2}{[1 + (\beta_1{}^2/\alpha^2)]}$$

$$\xi_2 = \alpha^2\beta_2\left(\frac{\beta_2{}^4}{\alpha^4} + \frac{\beta_2{}^2}{\alpha^2} - K^2\right) = -\frac{2K^2\beta_2\alpha^2}{[1 + (\beta_2{}^2/\alpha^2)]}$$

$$\xi_3 = \alpha^2\beta_3\left(-\frac{\beta_3{}^4}{\alpha^4} + \frac{\beta_3{}^2}{\alpha^2} + K^2\right) = \frac{2K^2\beta_3\alpha^2}{[1 - (\beta_3{}^2/\alpha^2)]} \tag{A15}$$

$$\eta_1 = \alpha\beta_1{}^2\left(1 + \frac{\beta_1{}^2}{\alpha^2}\right) \qquad \eta_2 = \alpha\beta_2{}^2\left(1 + \frac{\beta_2{}^2}{\alpha^2}\right)$$

$$\eta_3 = \alpha\beta_3{}^2\left(\frac{\beta_3{}^2}{\alpha^2} - 1\right) \qquad \zeta_1 = \alpha\beta_1\left(1 + \frac{\beta_1{}^2}{\alpha^2}\right)$$

$$_2 = \alpha\beta_2\left(1 + \frac{\beta_2{}^2}{\alpha^2}\right) \qquad \zeta_3 = \alpha\beta_3\left(1 - \frac{\beta_3{}^2}{\alpha^2}\right)$$

Equation (A8) has the solution

$$\{B_j\} = [g_{rj}]^{-1}\{k_r\}$$

Substitution into (A12) yields

$$\{l_i\} = [P_{ir}]\{k_r\} \tag{A16}$$

with

$$[P_{ir}] = [h_{ij}][g_{rj}]^{-1} \tag{A17}$$

Table 1 Values of $\bar{p}$[a]

	$(\bar{p})_1$		$(\bar{p})_2$		$(\bar{p}_3)$	
m, n	Dist. mass	Lumped mass	Dist. mass	Lumped mass	Dist. mass	Lumped mass
1, 2	0.8017	0.6007	1.936	1.074	5.006	2.591
1, 3	0.9520	0.9403	1.770	1.080	4.865	2.407
2, 2	0.8664	0.8774	4.931	1.481	5.152	2.592
2, 3	1.080	1.072	1.936	1.483	5.078	2.408
3, 2	1.298	1.265	2.190	1.759	5.131	2.600
3, 3	1.861	1.378	4.882	1.760	5.239	2.412

[a] $\bar{p} = \left(\frac{\rho\omega^2\bar{A}b^4}{E\bar{I}}\right)^{1/4}$.

Using the notation

$$c_r = P_{1r} \qquad r = 1, 2, \ldots, 6$$

$$c_7 = P_{22} \qquad c_8 = P_{23} \qquad c_9 = P_{25} \qquad c_{10} = P_{26}$$

$$c_{11} = P_{33} \qquad c_{12} = P_{36}$$

Equation (A16) may be written in the form of Eq. (1). Owing to the reciprocal relations[3] and the uniformity of the ring, there are only 12 independent coefficients c_i.

If

$$0.11340 < K^2 \leqslant 17.6366$$

the general solution of (A3) is of the form

$$V = B_1\,\mathrm{ch}\frac{\beta_1}{\alpha}\phi\cos\frac{\beta_2}{\alpha}\phi + B_2\,\mathrm{sh}\frac{\beta_1}{\alpha}\phi\cos\frac{\beta_2}{\alpha}\phi + B_3\,\mathrm{ch}\frac{\beta_1}{\alpha}\phi\sin\frac{\beta_2}{\alpha}\phi + B_4\,\mathrm{sh}\frac{\beta_1}{\alpha}\phi\sin\frac{\beta_2}{\alpha}\phi + B_5\cos\frac{\beta_3}{\alpha}\phi + B_6\sin\frac{\beta_3}{\alpha}\phi \tag{A18}$$

where $(\beta_1 \pm i\beta_2)^2/\alpha^2$, $-\beta_3{}^2/\alpha^2$ are the three values of Λ^2. Proceeding as before, it may be shown that Eq. (A17) still holds provided the elements of $[g_{ij}]$ and $[h_{ij}]$ are interpreted as follows:

$$\left.\begin{array}{l} g_{11} = 1 \qquad g_{12} = g_{13} = g_{14} = 0 \\ g_{15} = 1 \qquad g_{16} = 0 \\ g_{21} = 0 \qquad g_{22} = \dfrac{\beta_1}{\alpha} \qquad g_{23} = \dfrac{\beta_2}{\alpha} \\ g_{24} = g_{25} = 0 \qquad g_{26} = \dfrac{\beta_3}{\alpha} \\ g_{31} = \alpha + \dfrac{(\beta_1{}^2 - \beta_2{}^2)}{\alpha} \qquad g_{32} = g_{33} = 0 \\ g_{34} = \dfrac{2\beta_1\beta_2}{\alpha} \qquad g_{35} = \alpha - \dfrac{\beta_3{}^2}{\alpha} \qquad g_{36} = 0 \\ g_{41} = cc \qquad g_{42} = sc \\ g_{43} = cs \qquad g_{44} = ss \qquad g_{45} = \cos\beta_3 \\ g_{46} = \sin\beta_3 \qquad g_{51} = \dfrac{\beta_1 sc - \beta_2 cs}{\alpha} \end{array}\right\} \tag{A19}$$

[Equation (A19) continued on next page]

$$g_{52} = \frac{\beta_1 cc - \beta_2 ss}{\alpha} \qquad g_{53} = \frac{\beta_1 ss + \beta_2 cc}{\alpha}$$

$$g_{54} = \frac{\beta_1 cs + \beta_2 sc}{\alpha} \qquad g_{55} = -\frac{\beta_3}{\alpha}\sin\beta_3$$

$$g_{56} = \frac{\beta_3}{\alpha}\cos\beta_3$$

$$g_{61} = \left[\left\{\frac{\beta_1^2 - \beta_2^2}{\alpha} + \alpha\right\} cc - \frac{2\beta_1\beta_2}{\alpha} ss\right]$$

$$g_{62} = \left[\left\{\frac{\beta_1^2 - \beta_2^2}{\alpha} + \alpha\right\} sc - \frac{2\beta_1\beta_2}{\alpha} cs\right] \qquad \text{(A19)}$$

$$g_{63} = \left[\left\{\frac{\beta_1^2 - \beta_2^2}{\alpha} + \alpha\right\} cs + \frac{2\beta_1\beta_2}{\alpha} sc\right]$$

$$g_{64} = \left[\left\{\frac{\beta_1^2 - \beta_2^2}{\alpha} + \alpha\right\} ss + \frac{2\beta_1\beta_2}{\alpha} cc\right]$$

$$g_{65} = \left[\alpha - \frac{\beta_3^2}{\alpha}\right]\cos\beta_3,$$

$$g_{66} = \left[\alpha - \frac{\beta_3^2}{\alpha}\right]\sin\beta_3$$

$$cc = \cosh\beta_1 \cos\beta_2 \qquad cs = \cosh\beta_1 \sin\beta_2$$

$$sc = \sinh\beta_1 \cos\beta_2 \qquad ss = \sinh\beta_1 \sin\beta_2$$

The elements of $[h_{ij}]$ are as follows:

$$h_{11} = 0 \qquad h_{12} = -\beta_1\phi_1 + \beta_2\phi_2$$

$$h_{13} = -\beta_2\phi_1 - \beta_1\phi_2$$

$$h_{14} = h_{15} = 0 \qquad h_{16} = -\beta_3[(\beta_3^4/\alpha^2) - \beta_3^2 - K^2\alpha^2]$$

$$h_{21} = \phi_5 \qquad h_{22} = h_{23} = 0 \qquad h_{24} = \phi_6$$

$$h_{25} = \alpha\beta_3^2[(\beta_3^2/\alpha^2) - 1] \qquad h_{26} = 0$$

$$h_{31} = 0 \qquad h_{32} = -\beta_1\phi_3 + \beta_2\phi_4$$

$$h_{33} = -\beta_2\phi_3 - \beta_1\phi_4 \qquad h_{34} = h_{35} = 0$$

$$h_{36} = -\alpha\beta_3[1 - (\beta_3^2/\alpha^2)]$$

$$h_{41} = (\beta_1 sc - \beta_2 cs)\phi_1 - (\beta_1 cs + \beta_2 sc)\phi_2$$

$$h_{42} = (\beta_1 cc - \beta_2 ss)\phi_1 - (\beta_1 ss + \beta_2 cc)\phi_2$$

$$h_{43} = (\beta_1 ss + \beta_2 cc)\phi_1 + (\beta_1 cc - \beta_2 ss)\phi_2$$

$$h_{44} = (\beta_1 cs + \beta_2 sc)\phi_1 + (\beta_1 sc - \beta_2 cs)\phi_2 \qquad \text{(A20)}$$

$$h_{45} = \beta_3 \sin\beta_3[-(\beta_3^4/\alpha^2) + \beta_3^2 + K^2\alpha^2]$$

$$h_{46} = \beta_3 \cos\beta_3[(\beta_3^4/\alpha^2) - \beta_3^2 - K^2\alpha^2]$$

$$h_{51} = -\phi_5 cc + \phi_6 ss \qquad h_{52} = -\phi_5 sc + \phi_6 cs$$

$$h_{53} = -\phi_5 cs - \phi_6 sc \qquad h_{54} = -\phi_5 ss - \phi_6 cc$$

$$h_{55} = -\alpha\beta_3^2 \cos\beta_3[(\beta_3^2/\alpha^2) - 1]$$

$$h_{56} = -\alpha\beta_3^2 \sin\beta_3[(\beta_3^2/\alpha^2) - 1]$$

$$h_{61} = (\beta_1 sc - \beta_2 cs)\phi_3 - (\beta_1 cs + \beta_2 sc)\phi_4$$

$$h_{62} = (\beta_1 cc - \beta_2 ss)\phi_3 - (\beta_1 ss + \beta_2 cc)\phi_4$$

$$h_{63} = (\beta_1 ss + \beta_2 cc)\phi_3 + (\beta_1 cc - \beta_2 ss)\phi_4$$

$$h_{64} = (\beta_1 cs + \beta_2 sc)\phi_3 + (\beta_1 sc - \beta_2 cs)\phi_4$$

$$h_{65} = \alpha\beta_3(\sin\beta_3)[(\beta_3^2/\alpha^2) - 1]$$

$$h_{66} = \alpha\beta_3(\cos\beta_3)[1 - (\beta_3^2/\alpha^2)]$$

with

$$\phi_1 = (\beta_1^2 - \beta_2^2)^2/\alpha^2 - (4\beta_1^2\beta_2^2/\alpha^2) + \beta_1^2 - \beta_2^2 - K^2\alpha^2$$

$$\phi_2 = [4\beta_1\beta_2(\beta_1^2 - \beta_2^2)/\alpha^2] + 2\beta_1\beta_2$$

$$\phi_3 = \alpha^2[(\beta_1^2 - \beta_2^2)/\alpha^3) + 1] \qquad \phi_4 = 2\beta_1\beta_2/\alpha$$

$$\phi_5 = [(\beta_1^2 - \beta_2^2)^2 - 4\beta_1^2\beta_2^2] + \alpha(\beta_1^2 - \beta_2^2)$$

$$\phi_6 = 2\beta_1\beta_2\{[2(\beta_1^2 - \beta_2^2)/\alpha] + \alpha\}$$

$$cc = \cosh\beta_1 \cos\beta_2 \qquad cs = \cosh\beta_1 \sin\beta_2$$

$$sc = \sinh\beta_1 \cos\beta_2 \qquad ss = \sinh\beta_1 \sin\beta_2$$

If $0 < K^2 \leqslant 0.1134$, the solution involves circular functions only and may be similarly investigated.

Torsional Vibrations of Ring

These are governed by

$$(GJ/a^2)(\partial^2\bar{\theta}/\partial\phi^2) - (EI_\phi/a^2)\bar{\theta} = \rho I_p(\partial^2\bar{\theta}/\partial t^2) \qquad \text{(A21)}$$

where G is the shear modulus, J the torsion constant, I_ϕ the moment of inertia about an axis in the plane of the ring, I_p the polar moment of inertia of the ring section, and $\bar{\theta}$ the angle of twist.

On assuming $\bar{\theta} = \bar{\Theta}(\phi)\ \exp i\omega t$ and substituting into (A21) there results

$$(d^2\bar{\Theta}/d\phi^2) + [(\rho I_p\omega^2a^2/GJ) - (EI_\phi/GJ)]\bar{\Theta} = 0$$

The general solution of which is

$$\bar{\Theta} = A\cos(q/\alpha)\phi + B\sin(q/\alpha)\phi \qquad \text{(A.22)}$$

with

$$q = [(\rho I\rho\omega^2a^2\alpha^2/GJ) - (EI_\phi\alpha^2/GJ)]^{1/2}$$

and arbitrary constants A and B.

Substitution of the boundary conditions (Fig. 3) for the span $(r - 1, s)$ to (r, s)

$$\bar{\Theta} = \bar{\Theta}_{r-1,\,s} \text{ at } \phi = 0 \qquad \bar{\Theta}_{rs} = \bar{\Theta} \text{ at } \phi = \alpha$$

and for the span (r, s) to $(r + 1, s)$

$$\bar{\Theta} = \bar{\Theta}_{rs} \text{ at } \phi = 0 \qquad \bar{\Theta} = \bar{\Theta}_{r+1,\,s} \text{ at } \phi = \alpha$$

results in the two equations

$$\bar{\Theta} = [(\text{cosec}q)\bar{\Theta}_{rs} - (\cot q)\bar{\Theta}_{r-1,\,s}]\sin(q\phi/\alpha) + \bar{\Theta}_{r-1,\,s}\cos(q\phi/\alpha) \qquad \text{(A.23)}$$

$$\bar{\Theta} = [(\text{cosec}q)\bar{\Theta}_{r+1,\,s} - (\cot q)\bar{\Theta}_{rs}]\sin(q\phi/\alpha) + \bar{\Theta}_{rs}\cos(q\phi/\alpha) \qquad \text{(A.24)}$$

Referring to Fig. 3, we have from Eq. (A23)

$$(d\bar{\Theta}/d\phi)_{\phi=\alpha} = \bar{T}_{rs}{}^L/GJ$$

and from Eq. (A24)

$$(d\bar{\Theta}/d\phi)_{\phi=0} = \bar{T}_{rs}{}^R/GJ$$

Further

$$\bar{T}_{rs}{}^R - \bar{T}_{rs}{}^L = \bar{T}_{rs}$$

Application of these three conditions leads to the final result

$$\bar{T}_{rs} = -(GJ/\alpha a)[c_{14}(\bar{\Theta}_{r+1,\,s} + \bar{\Theta}_{r-1,\,s}) - 2c_{13}\bar{\Theta}_{rs}] \qquad \text{(A.25)}$$

with

$$c_{13} = q\cot q,\ c_{14} = q\ \text{cosec}q$$

The expression for the shear force Q_{rs}' follows from the relation given in Love.[2] The bending couple does not influence the equation of equilibrium, cancelling identically at each node.

Flexural Vibrations at Right Angles to the Plane of the Ring

Assuming that the angle of twist is very small, the vibrations are governed approximately by

$$(\partial^4 u/\partial\phi^4) + (\rho A a^4/EI_\phi)(\partial^2 u/\partial t^2) = 0 \qquad (A.26)$$

where u is the displacement at right angles to the ring and I_ϕ the moment of inertia of the section about an axis in the plane of the ring. Equation (A26) is identical to that of the vibrations of a straight beam, and Eqs. (5) and (6) may be readily deduced. The only difference between the straight beam and the ring is that the shearing force is a function of the torsional stiffness of the ring, and torsional moments are induced (for further details see Love[2]). Strictly speaking, the uncoupling of the out of plane vibrations and torsional vibrations is not possible. Since the gridwork model can, in any event, only approximately represent motion involving primarily u displacements, the simplified equation given is felt to be sufficiently accurate, especially for small arc length/radius ratios.

Lumped Mass

When the ring is supposed massless, its deflection is governed by

$$(d^6V/d\phi^6) + 2(d^4V/d\phi^4) + (d^2V/d\phi^2) = 0 \qquad (A27)$$

together with the condition of inextensibility $W = dV/d\phi$. The general solution of (A27) is

$$V = B_1 + B_2\phi + B_3\cos\phi + B_4\sin\phi + B_5\phi\cos\phi + B_6\phi\sin\phi \qquad (A28)$$

One may now proceed as in the distributed mass formulation and obtain the relation (A17). The matrix (P_{ir}) in this case can be evaluated explicitly without great difficulty. Writing Eqs. (A16) in the form of Eq. (1), the c_i are given by Eqs. (26).

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Free Vibrations of Ring-Stiffened Conical Shells

V. I. Weingarten*
University of Southern California, Los Angeles, Calif.

Linear shell theory was used to develop an analysis for predicting the natural frequencies of ring-stiffened simply supported conical shells. An experimental investigation was performed in conjunction with this analysis. In the analysis, the longitudinal, circumferential, and rotatory inertia forces were assumed to be small in comparison with the radial inertia force. The previous assumption simplified the uncoupling of the equilibrium equations. A linear Donnell-type vibration equation was obtained for orthotropic conical shells. By finding an equivalent orthotropic shell, the free vibration characteristics of a ring-stiffened conical shell were determined. Application of the Galerkin method reduced the shell equations to the form of frequency determinant. Digital computer techniques were used to solve the resulting frequency determinant.

Nomenclature

A_n	= displacement coefficient, see Eqs. (5)
A_s	= stiffener area
B_n	= displacement coefficient, see Eqs. (5)
C_n	= displacement coefficient, see Eqs. (5)
D	= flexural stiffness of shell wall $[Eh^3/12(1-\nu^2)]$
E	= Young's modulus for isotropic material
E_x, E_ϕ	= Young's modulus for orthotropic material
$F_1(n)$–$F_5(n)$	= expressions defined by Eqs. (14–18)
f	= frequency (cps) $(f = \omega/2\pi)$
$G_1(n, m)$, $G_2(n, m)$, $G_3(n, m)$	= expressions defined by Eqs. (11–13)
G	= shear modulus
h	= thickness of the shell
h_ϕ	= effective thickness in circumferential direction
I_ϕ	= effective moment of inertia of ring and shell combination, see Eq. (21)
I	= imaginary part of the expression, see Eqs. (5)
I_0	= moment of inertia of the stiffener about its own centroid
i	= $(-1)^{1/2}$
k_1	= see Eq. (22)
k_2	= see Eq. (23)
K^4	= geometry parameter $[12\mu(x_1/h)^2]$
l	= slant length
l_s	= length between stiffeners
L_1–L_8	= differential operators
L_7^{-1}	= inverse operator
m, n	= integers
R_1, R_2	= radii of small and large end of truncated cone, respectively
s	= complex number $[\gamma + in\beta]$
t	= cone circumferential wave number
t	= time

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* Assistant Professor of Civil Engineering. Member AIAA.